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GEODETIC APPLICATIONS OF STATISTCAL
HYPOTHESIS TESTS

GEODETIC APPLICATIONS OF STATISTICAL HYPOTHESIS TESTS

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Preface

The science of statistics can be of great use to geodesists and photogrammetrists in reducing and analyzing data obtained from measurements. This thesis will explore the applications of statistical hypothesis tests in the fields of geodesy.

The data for examples have been taken from various studies in the field of geodetic science. In each case the source of the data is noted. Often the conclusions drawn by statistical tests will not agree with the conclusions drawn by the original experimenter. It is not the intent of this thesis to criticize the work of others, but only to give examples of how statistical tests may be used to guide geodesists in drawing conclusions from observed data.

The author gratefully acknowledges the guidance of Dr. Urho A. Uotila and the assistance and inspiration of his wife, Judy, in the preparation of this thesis.

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CHAPTER 1

INTRODUCTION

In geodesy and photogrammetry, large quantities of data are collected in the form of measurements such as angles, lengths, and gravity values. It is imperative that the relevant information contained in a mass of geodetic data be expressed by comparatively few values. To accomplish this task, geodesists, as scientists in other fields of physical and social science, have found many solutions through the science of statistics.

Statistics is the study of populations, variances, and methods of data reduction. Some statistical methods, notably the method of least squares, have found universal acceptance in geodesy. Other statistical methods are not so widely used. It is the purpose of this thesis to explore the applications of statistical hypothesis testing to the problems of geodesy.

1.1 Statistical Theory

A STATISTIC is a value calculated from an observed sample with a view to characterizing the population from which it is drawn.

Statistics which will be of value to geodesy are:

Population mean	μ
Sample mean	$\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$
True variance	σ^2

Variance estimate[1]	$s^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})}{n-1}$
Fisher's F Statistic	$F = \frac{s_1^2}{s_2^2}$
Pearsen's χ^2 Statistic	χ^2
Student's t Statistic	t
The standard error	σ
The estimate of standard error[1]	s

A statistic which, on the average, gives the right answer is said to be unbiased. If a statistic gives values which are concentrated more closely to the right value, the statistic is said to be efficient.

A widely used concept in statistical theory is a random variable. A random variable is a quantity which takes on a definite value at every point of a sample space. Geodetic measurements are considered to be random variables.[2]

1.2 Density and Distribution Functions

Assume that a sample space is such that each point of the sample space can be characterized by the value of a continuous variable, x, which can take on all values between $-\infty$ and $+\infty$. If the event, A is defined as the set of all sample points characterized by the inequality

$$x \leq x^o$$

where x^o is some fixed value. The cumulative probability distribution function, $F(x^o)$, and the probability density function, $\emptyset(x)$, are de-

[1]The letter m is commonly used in geodetic literature. In most statistical literature Greek letters represent true values and the Roman equivalent represents the statistical estimate of this value. For ease of understanding, the statistical convention will be followed.

[2]See discussion by J.L. Stearn (1964).

fined by

$$P(A) = F(x^o) = \int_{-\infty}^{x^o} \phi(x)\,dx.$$

The density function, $\phi(x)$, has the additional significance that

$$P(x^o \leq x \leq x^o + dx) = \phi(x^o)dx.$$

Since the total probability for a sample space must equal unity, $\phi(x)$ must include a normalization factor

$$F(\infty) = \int_{-\infty}^{+\infty} \phi(x)\,dx = 1.$$

The basic requirement for any probability density function is that this integral exists. A further requirement is that

$$\phi(x) \geq 0 \quad \text{for} \quad -\infty < x < +\infty .$$

There are many distributions and density functions. As an example the uniform density function is defined

$$\phi(x) = \frac{1}{2a} \quad \text{for} \quad -a \leq x \leq +a$$

and

$$\phi(x) = 0 \quad \text{otherwise.}$$

Figure 1 shows the corresponding probability density function, $\phi(x)$, and the cumulative probability distribution function, F(x), for this uniform density function.

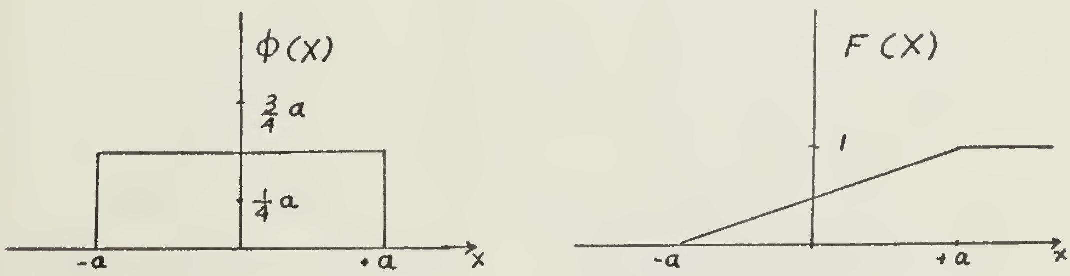

Figure 1

A quantity is said to be normally distributed when it takes all values from $-\infty$ to $+\infty$, with frequencies given by a definite mathematical law, namely, the logarithm of the frequency at any distance, d, from the center of the distribution is less than the logarithm of the frequency at the center by a quantity proportional to d^2. The distribution is symmetric with greatest frequency at the center (Fisher, 1925). The density function of a normal distribution of mean μ and variance σ^2 is given by the expression

$$\phi(X) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(X-\mu)^2}{2\sigma^2}} .$$

Figure 2 shows the probability density function and the cumulative probability distribution function for the normal distribution. The scale of x can be changed by measuring each x value by its distance from the mean, and adopting the standard deviation σ as a unit of measurement. An ordinate of this normal curve is then

$$w = \frac{(X-\mu)}{\sigma} .$$

The quantity w is called the standard normal deviate (Mandel, 1964). A unit normal deviate is a distribution which has a mean of 0, and a variance of unity.

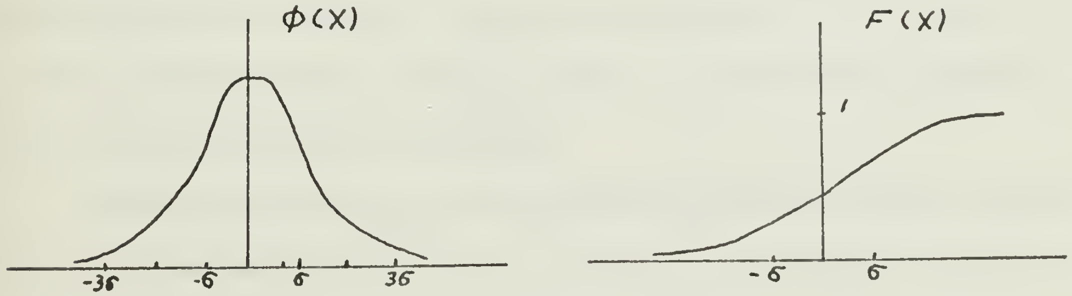

Figure 2

The normal distribution is very useful in statistics because of the Central Limit Theorem. This theorem may be expressed as follows:

Given a population of values with a finite varianee, if independent samples are taken from this population, all of size N, then the population formed by the averages of these samples will tend to have a normal distribution, regardless of what the distribution of the original population is; the larger N, the greater will be this tendency towards normality.

Thus far only functions of one variable have been discussed. If there are more than one variable associated with each point in sample space, multivariant funetions may be defined. For example, a multivariant probability density function is defined as

$$\phi(x_1^o, x_2^o, \ldots\ldots x_n^o)dx_1dx_2 \ldots\ldots dx_n$$
$$= P(x_1^o \leq x_i \leq x_i^o + dx_i,\ i = 1, 2, \ldots n).$$

The right side of this equation may be interpreted as the probability that all the inequalities hold simultaneously. For an excellent discussion, the reader is referred to pages 16 through 18 of Hamilton (1964).

1.3 Statistical Hypotheses

Webster defines a hypothesis as a tentative theory or supposition provisionally adopted to explain certain facts and to guide in the investigation of others (Webster, 1954). A statistical hypothesis is thus a theory about some population.

The only way that one can be absolutely certain of the truth or falsity of a statistical hypothesis is to examine the entire population. Since measurements can take on an infinite number of values, exami-

nation of the entire population in geodetic applications is impossible. One is then forced to make a decision based on a few measurements. Statistically speaking these measurements are a sample taken from the population. The process of using this sample to test the truth or falsity of a hypothesis is called statistical tests. There is in these tests no certainty that a mistake has not been made. There are, in fact, two different kinds of errors which can be made. These are called:

Type I (α) error -- the rejection of a hypothesis which is true

Type II (β) error -- the acceptance of a hypothesis which is false

These errors will be discussed in detail later.

Table 1

Definition of the Types of Errors Associated with the Tests of Statistical Hypotheses

	True Situation	
Decision	Hypothesis is True	Hypothesis is False
Accept the Hypothesis	No Error	Type II Error
Reject the Hypothesis	Type I Error	No Error

CHAPTER 2

HYPOTHESIS TESTING

It may be of value to know if data are normally distributed. A simple test is to compare the histogram to a normal curve (Dixon, 1957). The percentage of the data in a given group of the histogram can be compared to the area under the normal curve corresponding to the given group.

For example; given the mean of a data set as 30, and its standard deviation of 5, 80% of the data is found between 20 and 35. Is this data normally distributed?

The standard normal deviates for the data group are

$$X_{20} = 20\text{-}30 = -2.0$$

$$X_{35} = 35\text{-}30 = 1.0.$$

From Appendix I the area under the normal curve is

from $-\infty$ to 1.0 =		.8413
from $-\infty$ to -2	= 1.0 - .9772 =	.0228
The area under the normal curve =		.8185

or 81.85%. This would indicate that the data in this group has a distribution close to a normal distribution.

The χ^2 statistic, discussed in section 2.4, may also be used to test the distribution of a group of observations. A discussion of this test will be deferred until the statistic has been introduced.

For an example of more elaborate tests for normal distribution the reader is referred to the paper by Stearn (1964).

2.1 Tests Involving the Normal Distribution

If a single observation, x, is made from a normally distributed population of mean, μ, and variance, σ^2, statistical theory states that (Hamilton, 1964)

(1) $$w = \frac{(x - \mu)}{\sigma}$$

has a normal distribution. The probability that the magnitude of w calculated in this manner exceeds some specified value

(2) $$P(|w| > w_\gamma)$$

where w_γ is the value of w to the right of which lies an area γ under the probability density curve is

(3) $$P(|w| > w_\gamma) = \gamma.$$

We can then write

(4) $$P(|w| > w_\gamma) = \frac{1}{\sqrt{2\pi}}\left[\int_{-\infty}^{-w_\gamma} e^{-\frac{w}{2}}\,dw + \int_{w_\gamma}^{+\infty} e^{-\frac{w}{2}}\,dw\right].$$

From Figure 3 we can interpret the value of the integral as the cross hatched area under the curve.

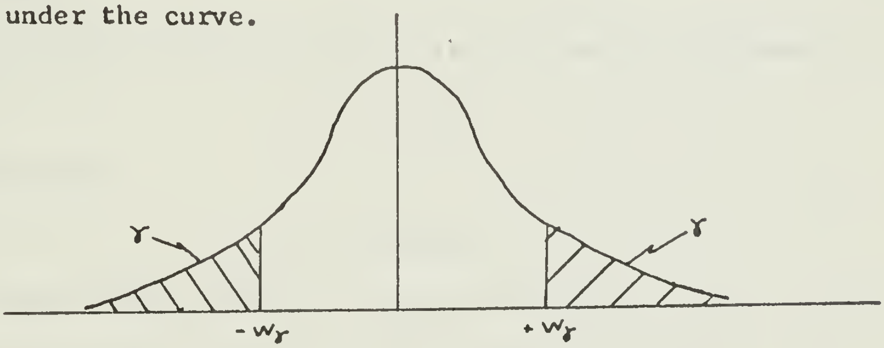

Figure 3

Mathematically Hamilton (1964), shows

(5) $$\begin{aligned} P(|w| > w_\gamma) &= 2F(-w_\gamma) \\ &= 2(1 - F(w_\gamma)) \\ &= 2\gamma \end{aligned}$$

The value of $F(w_\gamma)$ can be found from a table of the cumulative normal

distribution function (Appendix I).

If w_γ is 1.96

$$P(|w| > 1.96) = 2(1 - 0.975) = 0.05.$$

Equation (5) can be expressed as

$$(6) \qquad P\left(\left|\frac{x - \mu}{\sigma}\right| > w_\gamma\right) = 2[1 - F(w_\gamma)]$$

or

$$(7) \qquad P(\mu - \sigma w_\gamma < x < \mu + \sigma w_\gamma) = 1 - 2[1 - F(w_\gamma)]$$

$$= 2F(w_\gamma) - 1.$$

For the example

$$P(\mu - 1.96\sigma < x < \mu + 1.96\sigma) = 0.95$$

that is, the probability that a single observation, in the normal population given, will lie within 1.96σ of the mean, is 95%.

The inequality expressed in (6) forms the confidence interval which is the basis for the test of a hypothesis involving the mean of a normal population.

Symbolically this null hypothesis, H_0, can be expressed

$$(8) \qquad H_0 : \mu = \mu_0$$

and the alternative

$$H_1 : \mu \neq \mu_0.$$

The unit normal deviate is calculated from (1), if $|w| > 1.96$ the hypothesis, H_0, can be rejected. The risk of rejecting a true H_0 (TYPE I ERROR), or the <u>level of significance</u> (α) is .05. It should be noted that any other level of significance could be selected simply by selecting a different value for w_γ from the table. As an example, if w were 1.64, α would be .10.

The area γ, shown in Figure 3, varies directly as α. The

shaded area, 2γ is in this case equal to α . This region is known as the critical region (Ostle, 1963). If the value of the test statistic used in a particular test of a statistical hypothesis falls in this region, the hypothesis is rejected.

If $\alpha = 2\gamma$, the test is said to be two tailed; that is, the hypothesis was rejected either if

$$w > 1.96$$

or

$$w < -1.96.$$

The test can also be formulated such that the hypothesis is rejected only if

$$w > 1.96$$

or if

$$w < -1.96.$$

In each of these cases the probability of a Type I error (rejection of a true hypothesis) is the area under only one tail of the curve and

$$\alpha = \gamma \tag{9}$$

Figure (4) shows the critical region for the one-tailed test.

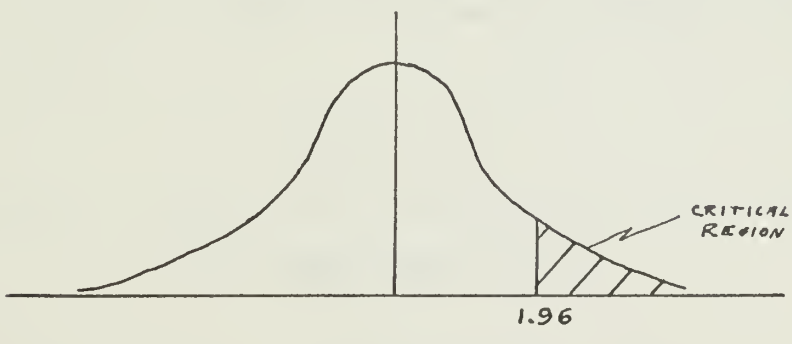

Figure 4

A test of a statistical hypothesis based on either extreme of the distribution is a one tailed test. It is interesting to note that for any symmetrical distribution a one tailed test at $100(\alpha)\%$ significance level is equivalent to a two tailed test at $100(2\alpha)\%$ significance level. This is true since the same critical value of w applies to the two cases.

2.11 Test Procedure

In order to test the hypothesis

$$H_o : \mu = \mu_o \qquad \emptyset = N(\mu, \sigma^2)$$

against the alternative

$$H_1 : \mu = \mu_o$$

at a significance level α, the following steps would be followed:

(1) Select a level of significance α, and find the corresponding w_γ from the appropriate table.

(2) Compute the unit normal deviate

$$w = \frac{X - \mu_o}{\sigma}$$

(3) Reject the hypothesis if $|w| > w_\gamma$.

For example, a set of observations has a sample mean $\bar{X}$ of 1.50, and a variance, σ^2, of 0.25. Could the true mean, μ_o, be 2.00? The hypothesis to be tested is

$$H_o : \mu_o = 2.00$$

If α is selected to be 0.05, the tabulated value of w_γ is 1.96.

$$w = \frac{1.50 - 2.00}{.5} = \frac{.5}{.5} = -1$$

$$1 < 1.96$$

Therefore, the hypothesis can be accepted at the level of significance selected.

The testing of a hypothesis such as this has very little practical significance in geodesy, since the variance, σ^2, is seldom known.

This test has been discussed in great detail because the principles involved apply to all statistical tests, regardless of the statistic, or distribution used.

2.2 Testing with a Sample Mean

The sample mean of a set of observations is defined as

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

where n is the number of observations.

Statistical theory states that the sum of normally distributed random variables is normally distributed (Hamilton, 1964).

The density function of $\bar{X}$ is

$$\phi\,(\bar{X}) = \frac{1}{\sigma}\left(\frac{n}{2\pi}\right)^{\frac{1}{2}} \exp \frac{-n(\bar{X} - \mu)^2}{2\sigma^2} \,.$$

The standardized variable is then

$$w = \frac{(\bar{X} - \mu)n^{\frac{1}{2}}}{\sigma}$$

Following the same method used for an individual observation, the probability that the deviation of the sample mean exceeds a specific value is

$$P\left(\mu - \frac{\sigma w_\gamma}{n^{\frac{1}{2}}} < \bar{X} < \mu + \frac{\sigma w_\gamma}{n^{\frac{1}{2}}}\right) = 2F(w_\gamma) - 1$$

To test a hypothesis from a sample of n observations and a sample mean of $\bar{X}$,

$$H_0 : \mu = \mu_0$$

against

$$H_1 : \mu \neq \mu_0$$

compute

$$w = \frac{(\bar{X} - \mu_0)n^{\frac{1}{2}}}{\sigma}$$

The hypothesis is rejected at the $100\alpha\%$ level if $|w| > w_\gamma$.

With this test a confidence interval is established around $\bar{X}$. The probability that the true mean, μ, lies within this interval is

$$P\left(\bar{X} - \frac{\sigma w_{\alpha/2}}{n^{1/2}} < \mu < \bar{X} + \frac{\sigma w_{\alpha/2}}{n^{1/2}}\right).$$

This test would be useful to determine if a new set of observations is part of an established population with a mean μ, and a standard error of σ.

It has been shown that the probability of a Type I error is the level of significance chosen for the test. There is also the possibility of committing a Type II error, that is the error of accepting a false hypothesis.

The probability, β, of a Type II error is dependent upon the specific alternative hypothesis which is presumed to be true

$$H_1 : \mu = \mu_1.$$

The probability of a Type II error is

(10) $$= P(-w_{\alpha/2} < w_o < w_{\alpha/2}),$$

when H_1 is true. From (1)

$$w = \frac{(\bar{X} - \mu_1)n^{\frac{1}{2}}}{\sigma}.$$

If the alternate hypothesis is true, w_1 is distributed as the unit normal deviate, and

(11) $$w_1 = \frac{n^{\frac{1}{2}}(\bar{X} - \mu_o + \mu_o - \mu_1)}{\sigma}$$

$$w_1 = w_o + \frac{(\mu_o - \mu_1)n^{\frac{1}{2}}}{\sigma}.$$

This can be written as

(12) $$w_o = w_1 + \frac{(\mu_1 - \mu_o)n^{\frac{1}{2}}}{\sigma}.$$

Equation (10) can then be written as

(13) $$\beta = P\left(-w_{\alpha/2} < w_1 + \frac{(\mu_1 - \mu_o)n^{\frac{1}{2}}}{\sigma} < w_{\alpha/2}\right)$$

$$(14)\quad \beta = P\left(-w_{\alpha/2} - \frac{(\mu_1 - \mu_0)n^{\frac{1}{2}}}{\sigma} < w_1 < w_{\alpha/2} - \frac{(\mu_1 - \mu_0)n^{\frac{1}{2}}}{\sigma}\right)$$

It should be noted that as $(\mu_0 - \mu_1)$ becomes smaller the probability of not rejecting a hypothesis when it is false is nearly as great as not rejecting it if it is true. The power of a statistical test is defined as $1 - \beta$. The power of a test increases as $\frac{\mu_0 - \mu_1}{\sigma}$ becomes larger or, as the number of observations increases (Hamilton, 1964). Figure 5 shows the power of the test for $\alpha = .05$ and $.01$, as a function of δ where $\delta = \frac{(\mu_1 - \mu_0)n^{\frac{1}{2}}}{\sigma}$.

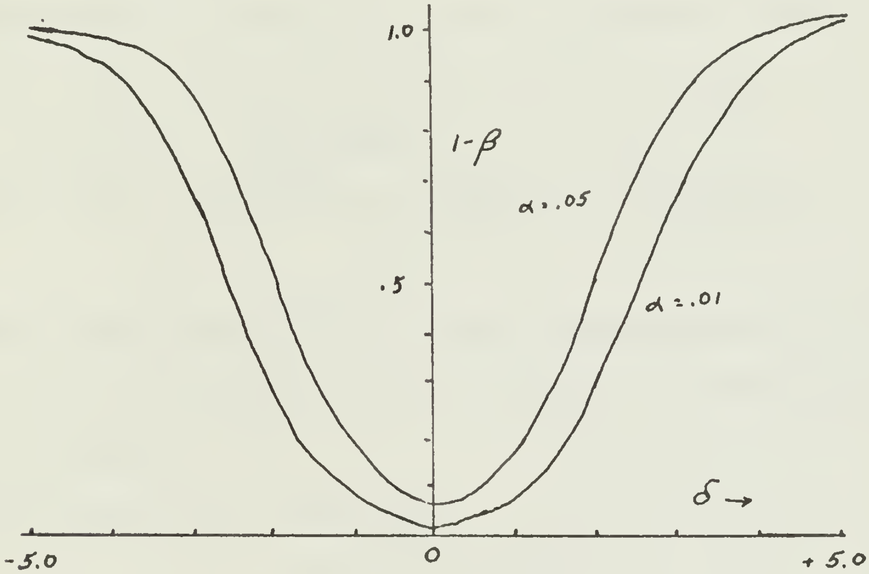

Figure 5

Note that $1 - \beta$ varies directly as α. The value of β is determined by the value chosen for α. If the critical region is increased, β decreases. Concerning this problem Graybill (1961) states,

> "We should like to minimize the likelyhood, or probability, of making either of these two errors. However, in general, for a fixed number of observations, if we decrease the probability of making an error of one type, we increase the chances of making the other."

Power curves can be found in Table A-12 of Dixon (1957).

Using the t distribution the test

$$H_o : \mu = \mu_o$$

against

$$H_1 : \mu \neq \mu_o$$

can be carried out.

First compute

$$t = \frac{(\bar{X} - \mu_o)n^{\frac{1}{2}}}{s_o} = \frac{(\bar{X} - \mu_o)}{s_{\bar{X}}} .$$

Reject the hypothesis at the α level of significance if

$$|t| > t_{n-1,\alpha} .$$

This test is valuable to the geodesist attempting to determine if a new set of observations is from the same population as previous observations with a mean of $\bar{X}$ and a variance estimate s^2. This problem often arises in the weighting of observations in an adjustment.

The following problem will serve as an example.

In his evaluation of the Laser Theodolite, Dunn (1966) observed the horizontal angles between two targets at various current levels. At 35 μ amps the angles obtained were (Dunn, p 52):

Left	Right	Mean	Residual
3° 17' 22".2	3° 06' 29".8	3° 06' 56".0	-5.0
07 17 .5	06 36 .8	06 57 .1	-3.9
07 00 .3	06 43 .6	06 52 .0	-9.0
08 29 .2	06 39 .8	07 34 .5	33.5
07 18 .4	06 33 .8	06 56 .1	-4.9
07 13 .1	06 27 .8	06 50 .4	-10.6
mean 3° 07' 26".8	3° 06' 35".3	3° 07' 01".0	

$[vv] = 1379.83 \qquad \frac{[vv]}{n-1} = s_o^2 = 276.0 \qquad s_o = \pm 16''.5$

The t test may be used to compare the mean of one set with a grand mean of previous sets.

From the previous two sets of observations the mean is

20 μ amps	3° 06' 51".1
30 μ amps	3° 06' 52".1
mean of 1 & 2	3° 06' 51".6

Test the hypothesis

$$H_o : \mu_{35} = 30\ 06'\ 51''.6$$

against

$$H_1 : \mu_{35} \neq 3^\circ\ 06'\ 51''.6$$

$$t = (3^\circ\ 07'\ 01''.0 - 3^\circ\ 06'\ 51''.6)\ (\sqrt{6})$$

$$t = \frac{9''.4\ (2.45)}{16.5} = 0.990$$

$$t_{5,\ 5\%} = 2.57.$$

Since $t < t_{5,\ 5\%}$, this hypothesis can be accepted at the 5% significance level.

2.4 Tests Involving Variance

The statistic χ^2 was introduced by Pearson. It is defined by the density function

(17) $$\phi(\chi^2) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} e^{-\chi^2/2} (\chi^2)^{\nu/2-1}$$

for $\chi^2 \geq 0$ otherwise $\phi(\chi^2) = 0$.

It is also known that the density function for the variance estimate s^2 from a sample size n of a population with a true variance σ^2 is (Hamilton, 1964)

$$\phi(s^2) = \frac{1}{\Gamma(\nu/2)} \left(\frac{\nu}{2\sigma^2}\right)^{\nu/2} \exp\left[-\frac{\nu s^2}{2\sigma^2}\right] (s^2)^{\nu/2-1}$$

for $s^2 > 0$ otherwise $\phi(s^2) = 0$.

Setting $\chi^2 = \frac{\nu s^2}{\sigma^2}$ these two density functions are identical. The value of $\frac{\nu s^2}{\sigma^2}$ is distributed as χ^2.

The value of χ^2 such that

$$P(\chi^2_\nu > \chi^2_{\nu,\alpha}) = \alpha$$

are tabulated in Appendix III.

The χ^2 statistic can be used to test whether the true variance estimated by s^2 is equal to some variance σ_0^2. The hypothesis can be stated

$$H_0 : \sigma^2 = \sigma_0^2$$

$$H_1 : \sigma^2 \neq \sigma_0^2 .$$

To test, compute $\chi^2 = \frac{\nu s^2}{\sigma_0^2}$

The hypothesis is rejected if

$$\chi^2 < \chi^2_{\nu,1-\alpha/2}$$

or

$$\chi^2 > \chi^2_{\nu,\alpha/2}$$

The use of this statistic to test hypothesis concerning geodetic and photogrammetric data is very limited. To compute the statistic, σ^2 the true variance of the population must be known. This quantity is rarely known for the types of data analyzed in geodesy.

One application of the Chi-square test would be in triangulation. The square of the desired standard error of the net could be considered as the true variance, σ^2. The Chi-square statistic computed at each station in the net, at the time of observation, could then be used as a criterion for acceptance or rejection of the observations.

The χ^2 statistic can be used as a test of distribution. To test the distribution, observations are grouped into n groups according to their values. The expected number of observations in each group is computed on the basis of the assumed distribution. Let n_k be the number of observations actually found in the kth group and d_k be the number predicted by the assumed distribution. The statistic

$$\chi^2 = \sum_{k=1}^{n} \frac{(n_k - d_k)^2}{d_k}$$

is distributed approximately as $\chi^2_{(n-1)}$ if each d_k is at least 5 (Hamilton, 1964).

2.5 The F Statistic

The F statistic or variance ratio, first introduced by Fisher, is a useful statistic.

F is defined

(18) $$F_{\nu_1 \nu_2} = (Y_1/\nu_1)/(Y_2/\nu_2)$$

where Y_1 is distributed as χ^2 with ν_1 degrees of freedom, and Y_2, INDEPENDENT of Y_1, is distributed as χ^2 with ν_2 degrees of freedom.

The probability density function is given as

$$\phi(F) = \frac{\Gamma([\nu_1 + \nu_2]/2)}{\Gamma(\nu_1/2)\Gamma(\nu_2/2)} \left(\frac{\nu_1}{\nu_2}\right)^{\frac{\nu_1}{2}} F^{\left(\frac{\nu_1 - 2}{2}\right)} \left(1 + \frac{\nu_1}{\nu_2}F\right)^{-\frac{\nu_1 + \nu_2}{2}}$$

for $F > 0$ otherwise $\phi(F) = 0$.

It was shown previously that

$$\chi^2 = \frac{\nu_1 s_1^2}{\sigma_1^2}$$

For two samples from two normal populations $\frac{s_1^2}{s_2^2}$ is distributed as

$$\frac{\sigma_1^2}{\sigma_2^2} F_{\nu_1, \nu_2}$$

Thus the hypothesis that the samples were drawn from populations with identical variances

$$H_0 : \sigma_1^2 = \sigma_2^2$$

can be tested against

$$H_1 : \sigma_1^2 \neq \sigma_2^2$$

compute

$$F = s_1^2/s_2^2 \quad .$$

The null hypothesis is rejected if the computed statistic falls in the selected critical region at either end of the distribution

$$F > F_{\nu 1, \nu 2, 1 - \alpha/2} \qquad \text{or}$$

$$F < F_{\nu 1, \nu 2, \alpha/2} \qquad \text{(Dixon, 1957).}$$

The inequalities shown above depend upon the definition of α . In some literature they will appear reversed.

Tables of the percentiles of the F distribution can be found in handbooks and statistics texts such as Dixon and Massey, Table A-7c. Appendix IV shows a sample of such a table.

As an example of the F test, the data taken by Dunn (1966) and Sprinsky can be analyzed.

Lt. Dunn made 34 observations of a horizontal angle with the Laser Theodolite.

The values were

3° 06'	56".1	
06	44 .7	
06	50 .4	
06	51 .6	
06	47 .2	
06	50 .6	
06	56 .2	
06	49 .0	
06	53 .9	
06	51 .1	
06	54 .6	
06	47 .7	
06	40 .5	
06	37 .7	
06	40 .2	
06	45 .5	
06	45 .8	
06	49 .7	
06	56 .0	
06	57 .1	
06	52 .0	
06	56 .1	
06	50 .4	
06	35 .0	
06	22 .4	
07	00 .1	
06	58 .4	
07	04 .3	
06	45 .3	
06	19 .6	
07	00 .7	
06	34 .8	
07	17 .6	
06	18 .2	

Mean of 34 observations 3° 06' 48".3

Standard deviation ± 12".4

Sample variance 153.8

Using the same equipment Capt. Sprinsky obtained the following

5°	27'	19"	.9
	26	58	.4
	27	00	.0
	26	53	.4
	26	54	.3
	26	57	.1
	27	13	.7
	27	22	.6
	26	58	.2
	26	41	.0
	27	07	.1
	27	13	.2
	26	48	.1
	26	41	.4
	26	30	.7
	26	44	.5
	26	49	.5
	26	45	.0
	26	52	.2
	26	33	.2
	26	38	.8
	26	39	.2
	26	38	.0
	26	28	.1
	26	40	.2
	26	29	.1
	27	10	.7
	26	42	.7

Mean of 28 observations 5° 26' 50".37

Standard deviation ± 14".3

Sample variance 204.5

Were the true variances of these two sets of observations equal? As a hypothesis this question can be stated

$$H_o : \sigma_1 = \sigma_2$$

and can be tested against the alternative

$$H_1 : \sigma_1 \neq \sigma_2$$

The test statistic is computed

$$F = \frac{204.5}{153.8} = 1.33$$

From a table of percentiles of F distribution, such as the sample in Appendix IV, the values of F are found to be

$$F_{28,\ 35,\ 2.5\%} = .489$$

$$F_{28,\ 35,\ 97.5\%} = 2.01\ .$$

Since

$$F > F_{28,\ 35,\ 2.5\%}$$

and

$$F < F_{28,\ 35,\ 97.5\%}\ ,$$

we can accept the hypothesis at the 5% significance level. It can then be concluded that the observations of Lt. Dunn and Capt. Sprinsky have the same true variance.

The following example will demonstrate another use of the F test.

In his study of the Kern DKM 3 theodolite Abby (1965) measured angles with a Wild T-3, and the DKM 3 using the center wire and the 5 wire field. These results were obtained:

	Instrument	Observations	Wires	s	s^2
1	T-3	16	1	0".99	0.980
2	T-3	16	1	.78	.608
3	DKM 3	15	5	.47	.221
4	DKM 3	10	5	.44	.194
5	DKM 3	10	1	.66	.436
6	DKM 3	30	5	.61	.372

The F statistic can be used to assist in drawing conclusions;

(a) Are the 5 wire observations significantly better than the 1 wire observations with the DKM 3? The hypothesis to be tested is

$$H_o : \sigma_1^2 \leq \sigma_2^2$$

For this example a one-tailed test will be used. The null hypothesis can be rejected if

$$\frac{s_1^2}{s_2^2} > F(1 - \alpha)(n_1 - 1)(n_2 - 1).$$

Comparing sets 3 and 5

$$F = \frac{0.436}{0.221} = 1.973$$

$$^F(14, 9)\ 5\% = 3.00$$

The difference is not significant at the 5% level.

Using sets 4 and 5

$$F = \frac{0.436}{0.194} = 2.247$$

$$^F(9, 9)\ 5\% = 3.18$$

The difference is not significant at the 5% level.

And using sets 5 and 6

$$F = \frac{0.436}{0.372} = 1.172$$

$$^F(15, 29)\ 5\% = 2.03.$$

Again the difference is not significant at the 5% level.

(b) Are the 5 wire, 30 observation sets significantly better than the 5 wire 10 observation sets?

$$F = \frac{0.372}{0.194} = 1.918$$

$$^F(29, 9)\ 5\% = 2.86$$

The difference is not significant.

(c) Is the DKM 3 significantly better than set 1 with the T-3?

$$F = \frac{.980}{.436} = 2.247$$

$$F_{(15,\ 15)\ 5\%} = 2.41$$

Again the test shows that the difference is not significant.

(d) Is the DKM 3, 5 wire procedure significantly better than the T-3?

$$F = \frac{0.980}{0.221} = 4.46 \quad \text{and} \quad F = \frac{.608}{.221} = 2.74$$

$$F_{(15,\ 14)\ 5\%} = 2.44$$

The test shows that the DKM 3, 5 wire procedure is significantly better than the T-3.

This problem also shows that statistical tests are not infallible. From parts (a) and (b) one can conclude that the difference in procedure is not significant. Part (c) draws the conclusion that the 1 wire DKM 3 procedure and the T-3 are not significantly different, yet (d) concludes that the 5 wire procedure is significantly different.

Statistical inference must be used with caution. Judgement must also be used in drawing conclusions.

2.6 Testing Correlations

The distribution of the sample correlation coefficient, r, is very complicated. Often in geodesy it is valuable to know if the true correlation coefficient, ρ, is zero. If ρ is equal to zero the statistic

$$\frac{r\sqrt{n-2}}{\sqrt{1-r^2}}$$

is distributed as Student's t with (n-2) degrees of freedom (Guttman, 1965).

To test the hypothesis

$$H_0 : \rho = 0$$

against

$$H_1 : \rho \neq 0$$

compute the statistic

$$C = \frac{r}{\sqrt{1-r^2}} \sqrt{n-2}$$

If $C > t_{n-2,\ \alpha/2}$, the hypothesis may be rejected in favor of the alternate.

To test the hypothesis

$$H_0 : \rho = \rho_1 \text{, where } \rho_1 \neq 0$$

we must know the distribution of ρ when $\rho \neq 0$. In most geodetic applications it is more important to know if correlation exists, rather than if it might be some specific value. For a discussion of the distribution of $\rho \neq 0$ the reader is referred to Graybill (1961).

CHAPTER 3

TESTS INVOLVING MULTIPLE VARIABLES

So far the tests which have been discussed deal only with one variable. The techniques of Regression Analysis allow the testing of hypothesis involving two or more related variables.

3.1 Regression Analysis

The functional relationship between some observable L and variables can be expressed mathematically

(19) $L = f(X_1, X_2 \ldots\ldots\ldots\ldots X_n / \theta_1, \theta_2 \cdots\cdots \theta_o)$

θ_i is a parameter in the function. This equation is often abbreviated

$$L = f(X_1, X_2 \ldots\ldots\ldots\ldots X_n).$$

To the statistician this equation is known as a regression function. The geodesist and photogrammetrist will recognize it as the mathematical structure for a method of observation equations adjustment problem.

The mathematical structure of the problem may be chosen by two methods. In geodesy the analytical consideration of the phenomenon involved is the preferred method. The examination of scatter diagrams plotted from the observed data can also yield a workable structure.

3.2 Fitting a Linear Mathematical Structure by Least Squares

Suppose that a linear relationship exists between a dependent

variable, y, and an independent variable, x, such as the relationship between gravity and height in free air. From observations n sets of points, (x_1, y_1), (x_2, y_2) ...... (x_n, y_n) are obtained. The sample structure can be expressed

$$y = \beta_0 + \beta_1 x$$

From the observed data we wish to obtain the values of the unknown β_0, β_1 and σ^2.

Let us assume that y for any given value of x is a random variable which is normally distributed with a mean $\beta_0 + \beta_1 x$ and variance σ^2. We will also assume that β_0 and β_1 do not depend upon x . The conditional probability density function of y for a given x can be written

$$(20) \quad f(y/x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-1}{2\sigma^2} (y - \beta_0 - \beta_1 x)^2\right].$$

The expected value of y, E(y/x) is

$$(21) \qquad E(y/x) = \beta_0 + \beta_1 x.$$

The conditional probability density function[3] of a set of n observations (x, y), (x_2, y_2) ..... (x_n, y_n) is

$$(22) \quad \prod_{i=1}^{n} f(y_i/x_i) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{1}{2\sigma^2}(y_i - \beta_0 - \beta_1 x_i)^2\right]$$

$$= \left\{\left(\frac{1}{2\pi\sigma^2}\right)^n \exp\left[-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i - \beta_0 - \beta_1 x_i)^2\right]\right\}$$

Applying the principles of maximum likelyhood (Guttman, 1965) the parameters which maximize the probability density function must be found. To maximize this function the quantity

$$(23) \qquad s = \sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i)^2$$

---

[3] For proof see Guttman (1965), Appendix III

must be minimized with respect to β_0 and β_1. The process of estimating β_0 and β_1 by minimizing the sum of the squares of the residuals is the well known <u>method of least squares</u>. The value of β_0 and β_1 which minimize $F(\beta_0, \beta_1)$ are those for which

$$\frac{\partial F(\beta_0 \beta_1)}{\partial \beta_0} = 0 \tag{24}$$
$$\frac{\partial F(\beta_0 \beta_1)}{\partial \beta_1} = 0.$$

It should be noted that for a normal distribution, the method of least squares gives a maximum likelyhood estimate of the parameters. If the error in the observed quantities is not normally distributed, the method of least squares will give a minimum variance estimate of the parameters but this estimate will <u>not</u> be the maximum likelyhood estimate.

Taking the derivitives of (23) we obtain

$$-2\sum_{i=1}^{n} (y_i - \beta_0 - \beta_1 x_i) = 0 \tag{25}$$
$$-2\sum_{i=1}^{n} x_i(y_i - \beta_0 - \beta_1 x_i) = 0$$

These equations can be rewritten as

$$\sum_{i=1}^{n} y_i = n\beta_0 + \beta_1 \sum_{i=1}^{n} x_i$$
$$\sum_{i=1}^{n} x_i y_i = \beta_0 \sum_{i=1}^{n} x_i + \beta_1 \sum_{i=1}^{n} x_i^2$$

In this form they are referred to as the normal equations. In matrix form this equation is the formula

$$NX + U = 0.$$

If the determinant of N is not zero a solution to this equation exists. The values are

$$b_0 = \overline{y} - b_1\overline{x}$$

$$b_1 = \frac{(x_i - \overline{x})(y_i - \overline{y})}{(x_i - \overline{x})^2}$$

where b_0 and b_1 are estimates of β_0 and β_1 (Guttman, 1965).

Graphically all values of x and y could be represented as a straight line in the x, y plane with intercept b_o, and slope b_1. This line is called the regression line.

The unknown parameters can be found using the matrix equation

$$NX = -U$$

or

$$X = -N^{-1}U$$

the variance of unit weight, s_o^2, (or m_o^2, as frequently used in geodetic literature) is given by

$$s_o^2 = \frac{V'PV}{n-u} \tag{28}$$

In the linear regression problem u is 2. From the matrix adjustment we also form the weight coefficient matrix, Q_X. The variance - covariance matrix Σ, is formed by multiplying Q_x by s_o^2. In the linear regression

$$\Sigma = \begin{bmatrix} s_{b_o}^2 & s_{b_o b_1} \\ s_{b_o b_1} & s_{b_1}^2 \end{bmatrix} \tag{29}$$

These quantities are the statistics which will be used in the testing of hypotheses concerning this regression.

3.3 Tests of Hypotheses in a Linear Regression

In most linear regressions the estimator of greatest importance is the slope of the line, b_1. To test if the slope is significantly different from some hypothesized value, say β_1' the t statistic is used. Stating the hypothesis

$$H_o : \beta_1 = \beta_1'$$

the alternate is

$$H_o : \beta_1 \neq \beta_1'$$

the statistic computed is

(30) $$t = \frac{(b_1 - \beta_1')}{s_1}$$

The hypothesis is rejected if

$$t \geq t_{(1- /2)} \quad (n-2)$$

or

$$t \leq -t_{(1- /2)} \quad (n-2)$$

It should be noted that if y is independent of x, β_1 will be 0. We can test for this by setting β_1' equal to 0, then

(31) $$t_i = \frac{b_i - 0}{s_{b_i}}$$

would be the test statistic.

Other hypotheses concerning the linear regression are

(1) $H_o : \beta_o = \beta_o'$

(2) $H_o : \beta_1 = \beta_1'$ and $\beta_o = \beta_o'$.

For hypothesis (1) the t statistic is used

(32) $$t = \frac{(b_o - \beta_o')}{s_{b_o}}$$

The F statistic can be used to simultaneously test hypothesis (2). This method will be discussed in section 3.31.

Table 2

Summary of Test Procedures

Hypothesis	Statistic	Equation	Rejection Region
$\beta_1 = \beta_1'$	t	30	$\lvert t \rvert \geq t_{(n-2)(1-\alpha/2)}$
$\beta_0 = \beta_0'$	t	32	$\lvert t \rvert \geq t_{(n-2)(1-\alpha/2)}$
$\beta_1 = \beta_1'$ and $\beta_0 = \beta_0'$	F	34 (sec. 3.31)	

The mathematical structure discussed is quite general. For example

$$y = \beta_0 + \beta_1 \sin t$$

can be handled by the methods previously discussed simply by substituting

$$x = \sin t.$$

The expotential problem $u = \gamma e^{\delta v}$ reduces to a linear form

$$\log u = \log \gamma = \delta v.$$

Making the following substitutions

$$y = \log u$$

$$\beta_0 = \log \gamma$$

$$x = v$$

$$\beta_1 = \delta$$

We have the problem in the form

$$y = \beta_0 + \beta_1 x \qquad \text{(Ostle, 1963).}$$

As one example of the geodetic applications of a linear regression data presented by Laurila (1965) can be analyzed. In this

report, Dr. Laurila assumes that the relationship between absolute humidity and measuring error is linear. The mathematical structure used is

$$D_t - D_o = CA + K$$

where D_o is the observed distance, D_t the true distance of the reference base, A denotes the absolute humidity, C represents the humidity coefficient, and K is an instrument constant.

From a least squares adjustment of 25 measurements the following results were obtained (Laurila, 1965).

C	s_c	K	s_k
0.97	±0.10	-11.4	±1.2

Figure 6 is a scatter diagram plotted from the data given in the report, for the reverse readings.

Using the t test, hypotheses about this line can be tested. It should be kept in mind that these tests are conditional tests. They take into account only the effect of errors of the parameter tested.

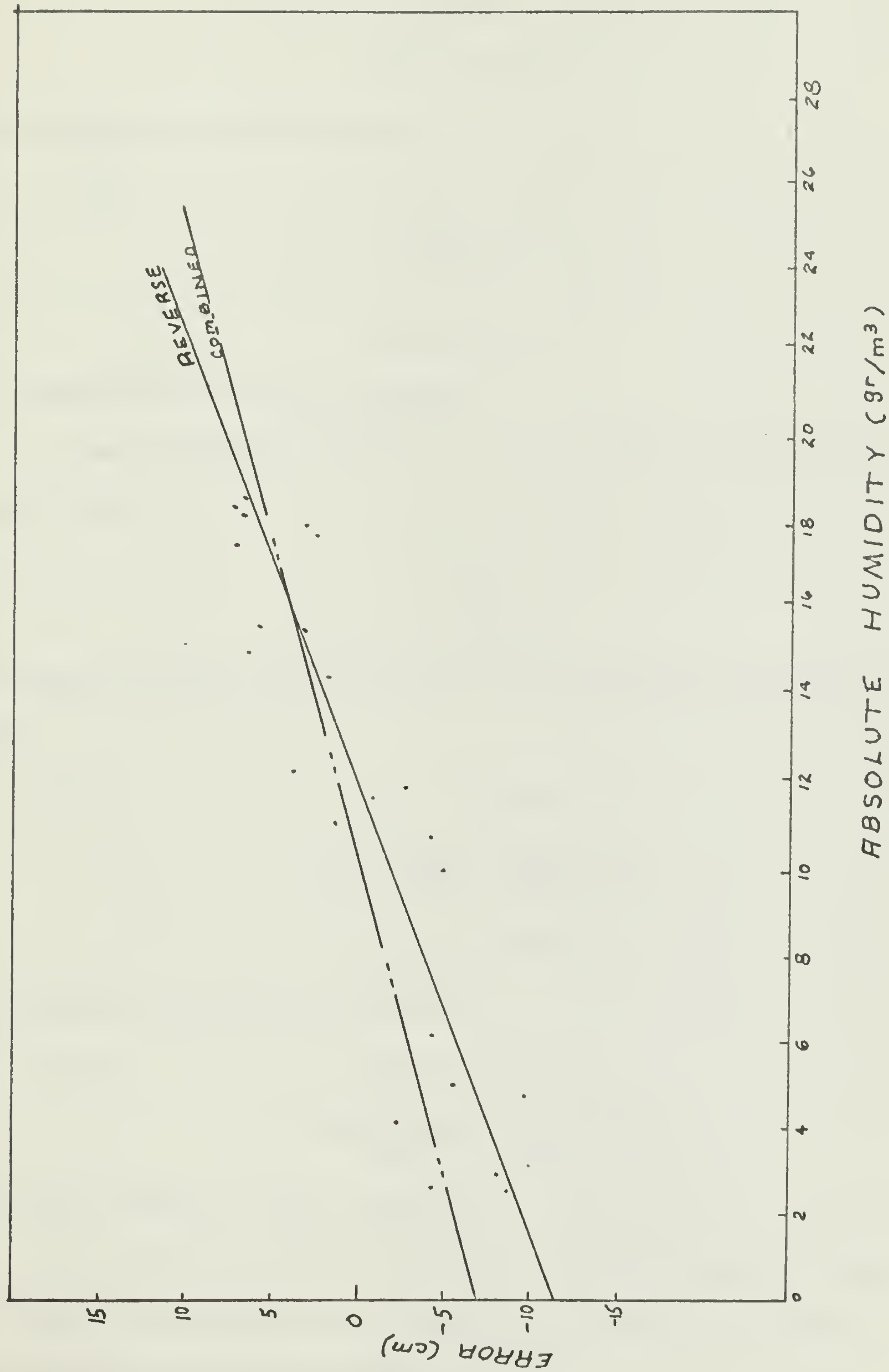
REVERSE
COMBINED
ABSOLUTE HUMIDITY (gr/m³)
ERROR (cm)
15
10
5
0
-5
-10
-15
0
2
4
6
8
10
12
14
16
18
20
22
24
26
28

Figure 6

(1) $$H_o : C = 0$$

$$t = \frac{0.97}{.10} = 9.7$$

$$t_{24,\ 5\%} = 2.06$$

This hypothesis can be rejected.

(2) $$H_o : K = 0$$

$$t = \frac{11.4}{1.2} = 9.5$$

$$t_{24,\ 5\%} = 2.06$$

This hypothesis can also be rejected.

From a combined adjustment (Laurila, 1965) the parameters were found to be

$$C = 0.66$$

$$K = -6.8.$$

Were these the true parameters for the line given by the forward readings only?

(3) $$H_o : C = 0.66$$

$$t = \frac{.97 - .66}{.10} = \frac{.31}{.10} = 3.1$$

$$t_{24,\ 5\%} = 2.06$$

This hypothesis may also be rejected.

(4) $$H_o : K = -6.8$$

$$t = \frac{11.4 - 6.8}{1.2} = \frac{4.6}{1.2} = 3.84$$

Again this hypothesis may be rejected.

In a similar manner the linear coefficients obtained from a least squares adjustment may be tested for a zero value or some theoretical or previously determined value.

3.31 Matrix Approach

The simplest way to handle adjustment computations is to use matrix algebra. It would be valuable to have matrix methods to test statistical hypothesis concerning the results of an adjustment. Hamilton (1964) has given such methods.

From an adjustment computation, using the methods outlined by Dr. Uotila (1966) the variance-covariance matrix Σ, the solution matrix X and the estimated standard error of one observation of unit weight s_o^2 can be obtained. From the derivation

$$\Sigma_x = s_o^2\, Qx = s_o^2\, N^{-1}$$

Hamilton (1964) shows that

$$(\bar{X} - X)'\Sigma^{-1}(\bar{X} - X)$$

is distributed as

$$\frac{\chi^2{}_u/u}{\chi^2{}_{n-u}/(n-u)} \equiv F_{u,\, n-u}$$

by matrix algebra

$$\Sigma = s^2 Q$$

$$\frac{Q^{-1}}{s^2} = \Sigma^{-1} = \frac{1}{s_o^2}(N).$$

Applying Hamilton's test

$$H_o : X = X_H$$

where X_H is the hypothesized value of X, compute

$$\frac{S_H}{u} = \frac{1}{u}(\bar{X} - X_H)'\,\frac{Q^{-1}}{s_o^2}(\bar{X} - X_H) \tag{34}$$

If $\frac{S_H}{u}$ exceeds $F_{u,\, n-u,\alpha}$ the hypothesis, H_o, may be rejected at the α level of significance.

Hamilton (1964) also developes the theory needed to test hypothesis when constraints have been placed on the structure.

Introducing Lagrange multipliers, K', the function

$$\phi = V'PV - 2K' (CX + Z)$$

is obtained. Differentiating this expression

$$d\phi = 2V'PdV - 2K'CdX.$$

Let $\overline{X}$ denote the least squares estimate under the constraint

$$d\phi = 2\left[-U + N\overline{X} - K'C\right] dX.$$

Minimizing the residuals

$$0 = 2\left[-U + \overline{X}'N - K'C\right] dX$$

so

$$K'C = -U' + \overline{X}'N$$

Substituting

$$U' = X^{*\prime}N$$

where X^* is the best least squares solution without conditions, the equation becomes

$$K'C = (\overline{X} - X^*)'N$$

$$KCN^{-1} C' = XC' - \overline{X}^{*\prime} C' = Z' - X^*C'$$

thus

$$K = (Z' - X^*C') (C^*N^{-1}C')^{-1}$$

eliminating K from the preceding two equations we obtain

$$(\overline{X} - X^*)'N = (Z - X^*C') (CN^{-1}C')^{-1}C$$

or

$$(35) \qquad \overline{X}' = X^{*\prime} + (Z - X^{*\prime}C') (CN^{-1}C')^{-1}CN^{-1}$$

The weighted sum of the squares of the residuals is given by

$$R_Q = V'PV + (\overline{X} - X^*)'(A'PA) (\overline{X} - X^*)$$

V is the matrix of residuals without the constraints.

The expected value of R_Q is $(n - u + b)\sigma^2$ where b is the number of conditions. The expression may be rewritten as

$$R_H = R_Q - R_o$$

where

$$R_o = V'PV$$

That is, R_o is the unconditional least squares sum and R_H is the additional sum of squares due to the constraints. Hamilton (1964) shows that

$$R_H = (Z - CX^*)' \, (C(A'PA)^{-1}C')^{-1} \, (Z - CX^*).$$

The ratio

$$\frac{R_H}{R_o} = \frac{R_Q - R_o}{R_o}$$

is distributed as

$$\frac{b}{n - u} \, F$$

Substituting the values of R_H and R_o

$$\frac{R_H}{R_o} = \frac{(Z - CX^*)' \, (C(A'PA)^{-1}C')^{-1} \, (Z - CX^*)}{(n - u)s^2},$$

but

$$\frac{A'PA}{s^2} = \frac{N}{s^2} = \Sigma^{-1}$$

the variance-covariance matrix.

Thus

$$(36) \qquad \frac{n - u}{b} \, \frac{R_H}{R_o} = \frac{(Z - CX^*)' \, (C\Sigma C')^{-1} \, (Z - CX^*)}{b}$$

which is distributed as $F_{b,\, n-u, \alpha}$. The hypothesis can be rejected if the computed value of this quantity exceeds the tabulated value of $F_{b,\, n-u, \alpha}$.

If a theoretical model has parameters X^T the hypothesis

$$H_o : X = X^T$$

that is

$$x_1 = x_1^T$$
$$x_2 = x_2^T$$
$$\vdots \quad \vdots$$
$$x_i = x_i^T$$

The matrices C and W are

$${}_uC_n = I \text{ and } {}_uZ_1 = X^T$$

thus the statistic to be tested is

$$\frac{(X^* - X^T)' \; Q_o^{-1} \; (X^* - X^T)}{us^2} = \frac{S_H}{u}$$

This is a more general derivation of the statistic offered earlier.

To test a hypothesis concerning the value of a single parameter, regardless of the values which the other parameters have, the same procedure would be followed.

$$H_o : x_1 = x_1^T$$
$$C = (1, 0, 0 \; ..... \; 0)$$
$$W = x_1^T$$

Then if the hypothesis is true

$$(n - u) \frac{R_H}{R_o} = \frac{(x_1^* - x_1^T)^2}{s_1^2}$$

where s_1^2 is the variance of x_1, the statistic $(n - u) \frac{R_H}{R_o}$ is distributed as $F_{1,\, n-u,\, \alpha}$. This is simply the square of the

Student's t for (n - u) degrees of freedom.

For a more detailed discussion of the matrix approach the reader is referred to chapter 4 of Hamilton (1964).

To illustrate the matrix approach, the data of Hatch (1964) will be used.

$$N = \begin{bmatrix} 15.00 & 0.00 & -0.10 & -0.01 & 0.01 & -0.20 & -0.03 \\ 0.00 & 7.66 & 0.13 & -0.20 & -0.01 & 0.02 & 0.03 \\ -0.10 & 0.13 & 7.42 & 0.05 & 0.00 & 0.02 & 0.01 \\ -0.01 & -0.20 & 0.05 & 7.31 & 0.00 & 0.03 & 0.00 \\ 0.01 & -0.01 & 0.00 & 0.00 & 0.03 & 0.00 & 0.00 \\ -0.20 & 0.02 & 0.02 & 0.03 & 0.00 & 2.32 & 0.00 \\ -0.03 & 0.03 & 0.01 & 0.00 & 0.00 & 0.00 & 0.27 \end{bmatrix}$$

Other data are

$$s = 2.44$$

$$s^2 = 5.95$$

$$u = 7$$

$$n = 15$$

$$X^* = \begin{bmatrix} 4.05 \\ -2.46 \\ 5.71 \\ 1.74 \\ 1.00 \\ -5.30 \\ -0.12 \end{bmatrix}$$

For the sake of an example, assume that some theoretical considerations predict X^T to be:

$$X^T = \begin{bmatrix} 3.66 \\ 0.00 \\ 6.83 \\ 0.00 \\ 0.00 \\ 0.36 \\ 0.00 \end{bmatrix}$$

We wish to test the hypothesis

$$H_o : X = X_T.$$

The computed values are

$$(X^* - X^T) = \begin{bmatrix} 0.39 \\ -2.46 \\ -1.12 \\ 1.74 \\ 1.00 \\ -5.66 \\ -0.12 \end{bmatrix}$$

$$\frac{S_H}{u} = \frac{(X^* - X^T)' \; Q^{-1} \; (X^* - X^T)}{us^2} = \frac{158.79}{41.65} = 3.81$$

From the tables for F (Appendix IV)

$$F_{7,\ 8,\ 5\%} = 3.50$$

Since $\frac{S_H}{u} < F_{7,\ 8,\ 5\%}$, the hypothesis may be rejected at

the 5% significance level. Rejection at this level of significance is termed <u>significant</u>. Can the hypothesis also be rejected at the 1%

level?

$$F_{7,\ 8,\ 1\%} = 6.84$$

$\frac{S_H}{u} < F_{7,\ 8,\ 1\%}$, so the hypothesis may not be rejected at the 1% significance level.

3.4 Complex Models

In many cases a simple two dimensional linear mathematical structure will not properly represent the given data. It may be that a polynomial of increasing order

$$Y = \beta_o + \beta_1 + \beta_2 x_1^2 + \ldots\ldots \beta_n x_1^n$$

will better represent the true structure. The matrix least squares solution in procedure is identical to the previous case. Using the solution matrix (b_o, b_1 ... b_n) and the weight coefficient matrix (Q) statistical hypothesis may be tested using the t tests previously shown.

The hypothesis

$$H_o : \beta_i = 0,\ (\beta_1 = \beta_1{}',\ \beta_2 = \beta_2{}' \cdots \beta_n = \beta_n{}')$$

is tested by computing

$$t = \frac{b_i^2}{s_o\sqrt{q_{ii}}} = \frac{b_i}{s_i}$$

and tested against

$$t_{(1 - \alpha/2)(n - u)}\ .$$

The hypothesis may also be tested by

$$F = \frac{b_i^2}{s_o^2 q_{ii}}\ .$$

These F tests serve to assess the significance of the additional reduction in the residual sum of squares achieved by fitting b's in the

particular order adopted. The order of fit is important. In a polynomial this order is fixed. In other structures the order is determined by the equations of the mathematical structure.

Then fitting polynomials Ostle (1963) recommends,

> "Rather than seek a better fit in terms of a higher degree polynomial (i.e. a degree greater than 2) it is probably better to cast about for some other functional form to represent the data."

An excellent example of regression analysis applied to a gravity problem is given in the article by H. Wolf (1965). In this paper Dr. Wolf uses hypothesis tests to determine the systematic trend of gravity differences along the European Calibration Line.

It is evident that the matrix approach discussed in Section 3.31 can be easily applied to complex models.

Application of hypothesis tests to a non-linear model raises another interesting problem. It cannot be shown that the non-linear least-squares solution will always converge to even a local minimum value of the weighted sum of the residuals, V'PV. This problem is discussed by Hamilton (1964). He concludes,

> "If the estimated errors are small enough that the functions are truly linear over the range of several standard deviations in each parameter, the methods of testing linear hypothesis ... can be applied in the same way."

When individual parameters are tested, either by the t or F tests previously demonstrated, any co-variance between the parameters is not considered. If the parameters were truly independent the choice between individual tests, i.e., the t test, and the simultaneous matrix tests would be one of individual preference.

If the co-variance between parameters is large, the simultaneous matrix test should be used.

Example: In his work with the MRA-1 Tellurometer, Hatch (1964) investigated the mathematical structure

$$T\text{-}B = b_1 + b_2 \sin\left(\frac{2\pi}{100}(A + b_5)\right) + b_3 \sin$$

$$+ \left(\frac{4\pi}{100}(A + b_6)\right) + b_4 \sin\left(\frac{6\pi}{100}(A + b_7)\right).$$

After least squares adjustment, results for January 19 were:

$$b(1) = 2.78 \pm 1.29$$

$$b(2) = 0.32 \pm 1.83$$

$$b(3) = 10.09 \pm 1.82$$

$$b(4) = -2.57 \pm 1.80$$

$$b(5) = -0.31 \pm 89.63$$

$$b(6) = 23.02 \pm 1.45$$

$$b(7) = -0.06 \pm 2.92$$

He made 16 observations thus

$$(n - u) = 16 - 7 = 9$$

degrees of freedom.

To test

$$H_o : \beta_i = 0$$

t_i is computed from $\quad t_i = b_i / s_{b_i}$

$$t_1 = \frac{2.78}{1.29} = 2.15$$

$$t_2 = \frac{0.32}{1.83} = 0.175$$

$$t_3 = \frac{16.09}{1.82} = 5.54$$

$$t_4 = \frac{2.57}{1.80} = 1.43$$

$$t_5 = 0.31 = 0.0035$$

$$t_6 = 23.02 = 15.90$$

$$t_7 = 0.060 = 0.0206$$

from the table

$$t_{9,\ 5\%} = 2.26,$$

thus the only parameters which are significant at the 5% level are b_3 and b_6.

At the 1% level

$$t_{9,\ 1\%} = 3.25$$

therefore b_1, b_3, and b_6 are significant at this level.

These tests at a significance level of 1% give statistical support to the conclusions drawn by Hatch (1964) that the error can be represented by

$$E = b_1 + b_3 \sin \frac{4\pi}{100}(A + b_6).$$

To go a step farther with the data given by Hatch, the hypothesis

$$H_o : B_i = 0,\ i = 1 \text{ to } 7$$

was tested on each set of observations using the t test. Table 3 shows the results (Hatch, 1964).

Table 3

Test of $H_o : B_i = 0$

Date/ Parameter	19 Jan.	21-22 Jan.	8 Feb.	25 March	15 April	17 April	Number of Rejections
B_1	A*	A	R	R	A	R	3
B_2	A	A	R	A	R	R	3
B_3	R	A	R	R	A	R	4
B_4	A	A	R	A	A	A	1
B_5	A	A	A	A	A	A	0
B_6	R	A	A	A	A	R	2
B_7	A	A	R	A	A	A	1

* A signifies acceptance, R signifies rejection.

If the parameters, which rejected the hypothesis more than once in the six sets are used, the formula for the cyclic zero error would be

$$E = b_1 + b_2 \sin \frac{2\pi}{100} A + b_3 \sin \frac{4\pi}{100} (A + b_6) \quad .$$

As a comparison of the individual t test and the simultaneous matrix test consider the following data taken on 23 March by Hatch (1964).

$$n = 25, \; u = 7, \; s_o = \pm 3.03, \; s_o^2 = 9.19$$

$$B(1) = 3.66 \pm 0.65$$

$$B(2) = -1.67 \pm 0.87$$

$$B(3) = 6.83 \pm 0.93$$

$$B(4) = 1.75 \pm 0.88$$

$$B(5) = 21.15 \pm 15.39$$

$$B(6) = 0.36 \pm 1.32$$

$$B(7) = 1.63 \pm 4.79$$

$$N = \begin{bmatrix} 25.00 & -2.95 & -0.26 & 2.58 & -0.27 & -0.05 & 0.11 \\ -2.95 & 13.97 & -2.82 & -2.23 & -0.02 & 1.60 & 0.17 \\ -0.26 & -2.28 & 11.62 & -1.58 & 0.01 & 0.18 & 0.14 \\ 2.58 & -2.23 & -1.58 & 13.05 & 0.01 & -1.92 & -0.11 \\ -0.27 & -0.02 & 0.01 & 0.01 & 0.04 & -0.08 & -0.01 \\ -2.05 & 1.60 & 0.18 & -1.92 & -0.08 & 6.16 & -0.28 \\ 0.11 & 0.17 & 0.14 & -0.14 & -0.01 & -0.28 & 0.42 \end{bmatrix}$$

The inverse matrix is given as:

$$N^{-1} = \begin{bmatrix} \underline{0.05} & 0.01 & 0.00 & -0.01 & 0.32 & 0.62 & -0.00 \\ 0.01 & \underline{0.08} & 0.02 & 0.01 & 0.05 & -0.02 & -0.00 \\ 0.00 & 0.02 & \underline{0.09} & 0.01 & -0.01 & -0.01 & -0.04 \\ -0.01 & 0.01 & 0.01 & \underline{0.03} & 0.00 & 0.02 & 0.03 \\ 0.32 & 0.05 & -0.01 & 0.00 & \underline{25.89} & 0.45 & 0.72 \\ 0.02 & -0.02 & -0.01 & 0.02 & 0.45 & \underline{0.19} & 0.14 \\ 0.00 & -0.05 & -0.04 & 0.03 & 0.72 & 0.14 & \underline{2.51} \end{bmatrix}$$

From the previous t tests one would expect that B(4), B(5), and B(7) might be equal to zero. Constraining all values to be equal to their test value, the C matrix is equal to I, and Z is equal to the test matrix, X^T. The hypothesis

$$H_o: X = X^T$$

where

$$X^T = \begin{bmatrix} 3.7 \\ -1.7 \\ 6.8 \\ 0.0 \\ 0.0 \\ 0.4 \\ 0.0 \end{bmatrix}$$

will be tested. Using equation (36)

$$\frac{S_H}{u} = \frac{(X^*-X^T)'N\,(X^*-X^T)}{u\,s_o} = \frac{44.697}{(7)(9.19)} = 0.695 \quad .$$

Since $\frac{S_H}{u} < F_{7,18,5\%} = 2.58$,

the hypothesis is acceptable. The values of B(4), B(5), and B(7) are probably equal to zero, which is the same conclusion reached by the t test. The values given for B(1), B(2), B(3), and B(6) are not necessarily the final values. The new structure, containing only these parameters, must be solved by another least squares adjustment. For the data used previously the results of this readjustment and the change from the values given by Hatch are

	Value	Change
B(1)	3.45	.21
B(2)	-1.80	-.13
B(3)	6.73	.10
B(6)	0.28	.16

Statistical tests could also be applied to these values.

The previous example could also have been tested using the "R factor" test discussed by Hamilton (1964). This is a test of the ratio $\frac{R_Q}{R_o}$. For the details the reader is referred to page 157 of Hamilton.

To point up the difference between the individual tests and a simultaneous test, the data of Laurila (1965), shown in Figure 6, Section 3.3 of this thesis will be used.

For this example assume that some theoretical consideration predicts that the values are

$$C' = 0.206$$

$$K' = 2.10 \quad .$$

Using the t test

$$t_{K'} = \frac{2.10}{1.2} = 2.06$$

$$t_{C'} = \frac{0.206}{0.10} = 2.06 \quad .$$

From the tables, $t_{24,5\%}$ is 2.06; therefore, these values of K' and C' are acceptable.

The values of K' and C' can then be used as the test values for a simultaneous F test.

$$(X^* - X^T) = \begin{bmatrix} 0.454 \\ -4.70 \end{bmatrix}$$

$$F = \frac{\begin{bmatrix} 0.454 & -4.7 \end{bmatrix} \begin{bmatrix} 25.0 & 272.4 \\ 272.4 & 3824.28 \end{bmatrix} \begin{bmatrix} 0.454 \\ -4.7 \end{bmatrix}}{(2)(0.2129)}$$

$$F = 195,683.$$

The tabulated value of $F_{7,18,5\%}$ is 2.58, thus the hypothesis can be rejected although the test values are acceptable on an individual basis. One would expect this, because the Q^{-1} matrix indicates a strong correlation between parameters.

CHAPTER 4

LEVEL OF SIGNIFICANCE

The selection of the significance level to be used in a statistical test is a matter of judgement and experience on the part of the geodesist. It may be chosen before the data is analyzed. Basically the problem is; how large should the critical region for rejection be, or what is the risk of committing a Type I error which the geodesist is willing to accept? Dixon (1957) states,

> "If it is a matter of great concern when a true hypothesis is rejected, α should be small ...... if it is a matter of great concern that a hypothesis be rejected if there is little evidence against it we should use a large α ".

A convention followed by statisticians is the following: If a hypothesis is rejected at $\alpha = 5\%$ it is said to be <u>significant</u>. A hypothesis rejected at $\alpha = 1\%$ is said to be <u>highly significant</u>.

When chosing a significance level for a test, one must keep in mind that the acceptance of a hypothesis is favored over rejection of an alternative by any test. If an α of 1% is chosen, the critical area for rejection is small, thus rejection is less likely than if a 5% level of significance is chosen.

The choice between $\alpha = 5\%$ and $\alpha = 1\%$ must be dictated by the circumstances surrounding the problem.

An entirely different approach to hypothesis testing can be developed by using a slightly different procedure. In this method the significance level is not pre-selected. The test statistic is computed by one of the methods previously discussed.

The probability (found in the tables) associated with the value

yielded by the data is taken as an objective measure of the degree of support that the data lends to the hypothesis. Taking this approach the test of significance does not lead to the acceptance or rejection of the hypotheses; it merely measures the strength of belief. In many problems encountered in geodesy and photogrammetry this may be the better approach.

Example: As a final step in his analysis Hatch (1964) represented each of his regression coefficients by a linear regression

$$b_1 = M_1\ e + c_1$$

$$b_2 = M_3\ P + C_3$$

$$b_6 = M_6\ e + C_6$$

Where e is the partial vapor pressure, and P is the barametric pressure.

By a least squares adjustment he obtained the following: (p. 66)

	Value	Standard Error
M_1	41.53	9.1
C_1	-8.08	2.0
M_3	-12.09	40.4
C_3	359.69	118.1
M_6	-144.82	30.8
C_6	43.80	6.8

The t statistic can be used to determine the significance of each co-efficient

$$t = \frac{x}{s_x}\ .$$

	t	Degree of Freedom	Degree of Support
M_1	4.57	4	98%
C_1	4.03	4	98%
M_3	.30	4	$\angle$ 50%
C_3	3.04	4	95%
M_6	4.71	4	99%
C_6	6.45	4	99%

It can be concluded that there is little relationship between b_2 and P since M_3 is significant at a level of less than 50%.

CHAPTER 5

CONCLUSIONS

This thesis has presented the statistical theory basic to hypothesis tests. Some of the statistics commonly used in hypothesis tests have been discussed, and their applications to the problems of geodesy demonstrated.

The tests most applicable to geodetic problems are obviously the ones which use a statistic that does not require the true variance in its computation. Thus the Student's t statistic and the F statistic are more useful.

The t statistic is used to test the mean of a population with an estimated standard error, s, and a finite number of observations, n, against some other mean value.

$$t = \frac{(\bar{X} - \mu_o)}{s_o} n^{\frac{1}{2}}$$

Fisher's F statistic is given as

$$F = \frac{s_1^{\,2}}{s_2^{\,2}} \;.$$

It can be used to test hypothesis concerning the variance of two independent sets of data.

From the adjustment procedures outlined by Dr. Uotila (1966), all the values needed to compute tests statistics are available. In most cases computations of the test statistic need only be carried to three significant figures. This precision can be obtained on an ordinary slide rule.

By using matrix methods the desired test statistic can be com-

puted, during the adjustment solution, by an electronic computer. The simultaneous matrix test is preferred over the individual t test in cases where there is a large correlation between parameters. The person analyzing the data need only compare the computed statistic with the tabulated values to make the hypothesis tests.

The purpose of using statistical hypothesis tests in geodesy and photogrammetry is to guide the user to the best conclusions based on the data analyzed. It should be emphasized that failure to reject a hypothesis does not mean the hypothesis is true. If, on the basis of a test, a hypothesis is rejected the statement can be made that there is evidence, from the data analyzed, that the hypothesis is not true. Conclusions drawn with the aid of statistical tests are thus supported by probability theory, in addition to the judgement and experience of the scientist.

APPENDIX I

CUMULATIVE NORMAL DISTRIBUTION

Appendix I

Cumulative Normal Distribution

Values of $Y = \int_{-\infty}^{X} \phi(x)\,dx$ for $X = 0.00[0.01]2.99$

$X \rightarrow$ ↓	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.5000	.5040	.5080	.5120	.5159	.5199	.5239	.5279	.5319	.5359
0.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
0.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
0.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
0.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
0.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
0.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
0.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
0.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
0.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986

Taken from Appendix, Table I (Hamilton, 1964).

APPENDIX II

STUDENT'S t DISTRIBUTION

Used in conjunction with problems in statistics, the table of the Student's t distribution permits the evaluation of deviations expressed in terms of estimates of standard errors for samples of various sizes. A given estimate of standard error is divided into a difference or deviation, to obtain t as a basis for a test of significance. The table is entered with the number of degrees of freedom determined for the problem. The tabular entry in the column is the value of t associated with the probability level indicated at the top of the column. This level expresses the probability of obtaining a difference as large as the one obtained due to chance. (Arkin and Colton 1959).

Student's t Distribution

Degrees of Freedom	Probability 0.50	0.10	0.05	0.02	0.01
1	1.000	6.34	12.71	31.82	63.66
2	0.816	2.92	4.30	6.96	9.92
3	.765	2.35	3.18	4.54	5.84
4	.741	2.13	2.78	3.75	4.60
5	.727	2.02	2.57	3.36	4.03
6	.718	1.94	2.45	3.14	3.71
7	.711	1.90	2.36	3.00	3.50
8	.706	1.86	2.31	2.90	3.36
9	.703	1.83	2.26	2.82	3.25
10	.700	1.81	2.23	2.76	3.17
11	.697	1.80	2.20	2.72	3.11
12	.695	1.78	2.18	2.68	3.06
13	.694	1.77	2.16	2.65	3.01
14	.692	1.76	2.14	2.62	2.98
15	.691	1.75	2.13	2.60	2.95
16	.690	1.75	2.12	2.58	2.92
17	.689	1.74	2.11	2.57	2.90
18	.688	1.73	2.10	2.55	2.88
19	.688	1.73	2.09	2.54	2.86
20	.687	1.72	2.09	2.53	2.84
21	.686	1.72	2.08	2.52	2.83
22	.686	1.72	2.07	2.51	2.82
23	.685	1.71	2.07	2.50	2.81
24	.685	1.71	2.06	2.49	2.80
25	.684	1.71	2.06	2.48	2.79
26	.684	1.71	2.06	2.48	2.78
27	.684	1.70	2.05	2.47	2.77
28	.683	1.70	2.05	2.47	2.76
29	.683	1.70	2.04	2.46	2.76
30	.683	1.70	2.04	2.46	2.75
35	.682	1.69	2.03	2.44	2.72
40	.681	1.68	2.02	2.42	2.71
45	.680	1.68	2.02	2.41	2.69
50	.679	1.68	2.01	2.40	2.68
60	.678	1.67	2.00	2.39	2.66
70	.678	1.67	2.00	2.38	2.65
80	.677	1.66	1.99	2.38	2.64
90	.677	1.66	1.99	2.37	2.63
100	.677	1.66	1.98	2.36	2.63
125	.676	1.66	1.98	2.36	2.62
150	.676	1.66	1.98	2.35	2.61
200	.675	1.65	1.97	2.35	2.60
300	.675	1.65	1.97	2.34	2.59
400	.675	1.65	1.97	2.34	2.59
500	.674	1.65	1.96	2.33	2.59
1000	.674	1.65	1.96	2.33	2.58
∞	.674	1.64	1.96	2.33	2.58

* The greater portion of this table taken from R. A. Fisher's "Statistical Methods for Research Workers," with the permission of the author and his publishers, Oliver and Boyd, London.

Source: Reproduced by permission from C. H. Goulden, *Methods of Statistical Analysis* (New York: John Wiley & Sons, 1939).

Taken from (Arkin and Colton 1959)

APPENDIX III

TABLE OF CHI-SQUARE

The table of Chi-square is entered with the degrees of freedom appropriate to the problem. The row for the specified degrees of freedom is followed across to the columns corresponding to $\alpha/2$ and $1 - \alpha/2$ where the theoretical values of Chi-square needed for the test are found.

Percentage Points of the χ^2 Distribution*†

Values of $\chi^2_{n,\alpha}$, where α is the probability that χ^2 exceeds $\chi^2_{n,\alpha}$, and

$$\int_0^{\chi^2_{n,\alpha}} \phi(\chi^2)\, d\chi^2 = 1 - \alpha \ ‡$$

n \ α	0.995	0.990	0.975	0.950	0.500	0.050	0.025	0.010	0.005
1	0.00+	0.00+	0.00+	0.00+	0.45	3.84	5.02	6.63	7.88
2	0.01	0.02	0.05	0.10	1.39	5.99	7.38	9.21	10.60
3	0.07	0.11	0.22	0.35	2.37	7.81	9.35	11.34	12.84
4	0.21	0.30	0.48	0.71	3.36	9.49	11.14	13.28	14.86
5	0.41	0.55	0.83	1.15	4.35	11.07	12.83	15.09	16.75
6	0.68	0.87	1.24	1.64	5.35	12.59	14.15	16.81	18.55
7	0.99	1.24	1.69	2.17	6.35	14.07	16.01	18.48	20.28
8	1.34	1.65	2.18	2.73	7.34	15.51	17.53	20.09	21.96
9	1.73	2.09	2.70	3.33	8.34	16.92	19.02	21.67	23.59
10	2.16	2.56	3.25	3.94	9.34	18.31	20.48	23.21	25.19
11	2.60	3.05	3.82	4.57	10.34	19.68	21.92	24.72	26.76
12	3.07	3.57	4.40	5.23	11.34	21.03	23.34	26.22	28.30
13	3.57	4.11	5.01	5.89	12.34	22.36	24.74	27.69	29.82
14	4.07	4.66	5.63	6.57	13.34	23.68	26.12	29.14	31.32
15	4.60	5.23	6.27	7.26	14.34	25.00	27.49	30.58	32.80
16	5.14	5.81	6.91	7.96	15.34	26.30	28.85	32.00	34.27
17	5.70	6.41	7.56	8.67	16.34	27.59	30.19	33.41	35.72
18	6.26	7.01	8.23	9.39	17.34	28.87	31.53	34.81	37.16
19	6.84	7.63	8.91	10.12	18.34	30.14	32.85	36.19	38.58
20	7.43	8.26	9.59	10.85	19.34	31.41	34.17	37.57	40.00
25	10.52	11.52	13.12	14.61	24.34	37.65	40.65	44.31	46.93
30	13.79	14.95	16.79	18.49	29.34	43.77	46.98	50.89	53.67
40	20.71	22.16	24.43	26.51	39.34	55.76	59.34	63.69	66.77
50	27.99	29.71	32.36	34.76	49.33	67.50	71.42	76.15	79.49
60	35.53	37.48	40.48	43.19	59.33	79.08	83.30	88.38	91.95
70	43.28	45.44	48.76	51.74	69.33	90.53	95.02	100.42	104.22
80	51.17	53.54	57.15	60.39	79.33	101.88	106.63	112.33	116.32
90	59.20	61.75	65.65	69.13	89.33	113.14	118.14	124.12	128.30
100	67.33	70.06	74.22	77.93	99.33	124.34	129.56	135.81	140.17

* Adapted from the tables prepared by Catherine M. Thompson for *Biometrika*, vol. 32; reproduced with permission of the editors of *Biometrika*.

† For more than 100 degrees of freedom, percentage points $\chi^2_{n,\alpha}$ of the χ^2 distribution may be obtained from the two-tailed percentage points X_P of the normal distribution by the approximate relation, $\chi^2_{n,\alpha} \approx n + (2n)^{1/2} X_P$, with $\alpha = P$.

‡ α is thus the probability in one tail of the distribution.

Taken from Appendix Table III (Hamilton, 1964)

APPENDIX IV

TABLE OF F

"In the table of F the distribution of F is tabulated for the 5% and 1% residual levels. The 5% level value of F is indicated in ordinary type, and the 1% level figures are printed in bold face type. Entry into the table is accomplished by reference to the appropriate column for the degrees of freedom associated with the greater variance, and to the appropriate row for the degrees of freedom associated with the smaller variance. If the calculated ratio between the two variances (F) exceeds the value for F indicated in the body of the table for the 5% level, there are fewer than 5 chances in 100 that the disparity between the calculated variances is due to chance; if F exceeds that recorded for the 1% level, the probability is less than 1 in 100 that the difference is accidental." (Arkin and Colton, 1959).

5% (Roman Type) and 1% (Bold Face Type) Points for the Distribution of F

n_2		n_1 degrees of freedom (for greater mean square) 1	2	3	4	5	6	7	8	9	10	11	12	14	16	20	24	30	40	50	75	100	200	500	∞	n_2
1	5%	161	200	216	225	230	234	237	239	241	242	243	244	245	246	248	249	250	251	252	253	253	254	254	254	1
	1%	**4,052**	**4,999**	**5,403**	**5,625**	**5,764**	**5,859**	**5,928**	**5,981**	**6,022**	**6,056**	**6,082**	**6,106**	**6,142**	**6,169**	**6,208**	**6,234**	**6,258**	**6,286**	**6,302**	**6,323**	**6,334**	**6,352**	**6,361**	**6,366**	
2	5%	18.51	19.00	19.16	19.25	19.30	19.33	19.36	19.37	19.38	19.39	19.40	19.41	19.42	19.43	19.44	19.45	19.46	19.47	19.47	19.48	19.49	19.49	19.50	19.50	2
	1%	**98.49**	**99.00**	**99.17**	**99.25**	**99.30**	**99.33**	**99.34**	**99.36**	**99.38**	**99.40**	**99.41**	**99.42**	**99.43**	**99.44**	**99.45**	**99.46**	**99.47**	**99.48**	**99.48**	**99.49**	**99.49**	**99.49**	**99.50**	**99.50**	
3	5%	10.13	9.55	9.28	9.12	9.01	8.94	8.88	8.84	8.81	8.78	8.76	8.74	8.71	8.69	8.66	8.64	8.62	8.60	8.58	8.57	8.56	8.54	8.54	8.53	3
	1%	**34.12**	**30.82**	**29.46**	**28.71**	**28.24**	**27.91**	**27.67**	**27.49**	**27.34**	**27.23**	**27.13**	**27.05**	**26.92**	**26.83**	**26.69**	**26.60**	**26.50**	**26.41**	**26.35**	**26.27**	**26.23**	**26.18**	**26.14**	**26.12**	
4	5%	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.93	5.91	5.87	5.84	5.80	5.77	5.74	5.71	5.70	5.68	5.66	5.65	5.64	5.63	4
	1%	**21.20**	**18.00**	**16.69**	**15.98**	**15.52**	**15.21**	**14.98**	**14.80**	**14.66**	**14.54**	**14.45**	**14.37**	**14.24**	**14.15**	**14.02**	**13.93**	**13.83**	**13.74**	**13.69**	**13.61**	**13.57**	**13.52**	**13.48**	**13.46**	
5	5%	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.78	4.74	4.70	4.68	4.64	4.60	4.56	4.53	4.50	4.46	4.44	4.42	4.40	4.38	4.37	4.36	5
	1%	**16.26**	**13.27**	**12.06**	**11.39**	**10.97**	**10.67**	**10.45**	**10.27**	**10.15**	**10.05**	**9.96**	**9.89**	**9.77**	**9.68**	**9.55**	**9.47**	**9.38**	**9.29**	**9.24**	**9.17**	**9.13**	**9.07**	**9.04**	**9.02**	
6	5%	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.03	4.00	3.96	3.92	3.87	3.84	3.81	3.77	3.75	3.72	3.71	3.69	3.68	3.67	6
	1%	**13.74**	**10.92**	**9.78**	**9.15**	**8.75**	**8.47**	**8.26**	**8.10**	**7.98**	**7.87**	**7.79**	**7.72**	**7.60**	**7.52**	**7.39**	**7.31**	**7.23**	**7.14**	**7.09**	**7.02**	**6.99**	**6.94**	**6.90**	**6.88**	
7	5%	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.63	3.60	3.57	3.52	3.49	3.44	3.41	3.38	3.34	3.32	3.29	3.28	3.25	3.24	3.23	7
	1%	**12.25**	**9.55**	**8.45**	**7.85**	**7.46**	**7.19**	**7.00**	**6.84**	**6.71**	**6.62**	**6.54**	**6.47**	**6.35**	**6.27**	**6.15**	**6.07**	**5.98**	**5.90**	**5.85**	**5.78**	**5.75**	**5.70**	**5.67**	**5.65**	
8	5%	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.34	3.31	3.28	3.23	3.20	3.15	3.12	3.08	3.05	3.03	3.00	2.98	2.96	2.94	2.93	8
	1%	**11.26**	**8.65**	**7.59**	**7.01**	**6.63**	**6.37**	**6.19**	**6.03**	**5.91**	**5.82**	**5.74**	**5.67**	**5.56**	**5.48**	**5.36**	**5.28**	**5.20**	**5.11**	**5.06**	**5.00**	**4.96**	**4.91**	**4.88**	**4.86**	
9	5%	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.13	3.10	3.07	3.02	2.98	2.93	2.90	2.86	2.82	2.80	2.77	2.76	2.73	2.72	2.71	9
	1%	**10.56**	**8.02**	**6.99**	**6.42**	**6.06**	**5.80**	**5.62**	**5.47**	**5.35**	**5.26**	**5.18**	**5.11**	**5.00**	**4.92**	**4.80**	**4.73**	**4.64**	**4.56**	**4.51**	**4.45**	**4.41**	**4.36**	**4.33**	**4.31**	
10	5%	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.97	2.94	2.91	2.86	2.82	2.77	2.74	2.70	2.67	2.64	2.61	2.59	2.56	2.55	2.54	10
	1%	**10.04**	**7.56**	**6.55**	**5.99**	**5.64**	**5.39**	**5.21**	**5.06**	**4.95**	**4.85**	**4.78**	**4.71**	**4.60**	**4.52**	**4.41**	**4.33**	**4.25**	**4.17**	**4.12**	**4.05**	**4.01**	**3.96**	**3.93**	**3.91**	
11	5%	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.86	2.82	2.79	2.74	2.70	2.65	2.61	2.57	2.53	2.50	2.47	2.45	2.42	2.41	2.40	11
	1%	**9.65**	**7.20**	**6.22**	**5.67**	**5.32**	**5.07**	**4.88**	**4.74**	**4.63**	**4.54**	**4.46**	**4.40**	**4.29**	**4.21**	**4.10**	**4.02**	**3.94**	**3.86**	**3.80**	**3.74**	**3.70**	**3.66**	**3.62**	**3.60**	
12	5%	4.75	3.88	3.49	3.26	3.11	3.00	2.92	2.85	2.80	2.76	2.72	2.69	2.64	2.60	2.54	2.50	2.46	2.42	2.40	2.36	2.35	2.32	2.31	2.30	12
	1%	**9.33**	**6.93**	**5.95**	**5.41**	**5.06**	**4.82**	**4.65**	**4.50**	**4.39**	**4.30**	**4.22**	**4.16**	**4.05**	**3.98**	**3.86**	**3.78**	**3.70**	**3.61**	**3.56**	**3.49**	**3.46**	**3.41**	**3.38**	**3.36**	
13	5%	4.67	3.80	3.41	3.18	3.02	2.92	2.84	2.77	2.72	2.67	2.63	2.60	2.55	2.51	2.46	2.42	2.38	2.34	2.32	2.28	2.26	2.24	2.22	2.21	13
	1%	**9.07**	**6.70**	**5.74**	**5.20**	**4.86**	**4.62**	**4.44**	**4.30**	**4.19**	**4.10**	**4.02**	**3.96**	**3.85**	**3.78**	**3.67**	**3.59**	**3.51**	**3.42**	**3.37**	**3.30**	**3.27**	**3.21**	**3.18**	**3.16**	

The function, $F = e$ with exponent $2z$, is computed in part from Fisher's table VI (7). Additional entries are by interpolation, mostly graphical.

Source: Reproduced from George W. Snedecor, *Statistical Methods* (Ames, Iowa: Iowa State College Press, 1946), pp. 222–225. By permission of the author.

Taken from (Arkin and Colton 1959).

TABLE A-7c. PERCENTILES OF THE $F(\nu_1, \nu_2)$ DISTRIBUTION WITH DEGREES OF FREEDOM ν_1 FOR THE NUMERATOR AND ν_2 FOR THE DENOMINATOR

ν_2	Cum. Prop. \ ν_1	1	2	3	4	5	6	7	8	9	10	11	12	Cum. Prop.
1	.0005	$.0^{6}62$	$.0^{3}50$	$.0^{2}38$	$.0^{2}94$	.016	.022	.027	.032	.036	.039	.042	.045	.0005
	.001	$.0^{5}25$	$.0^{2}10$	$.0^{2}60$	.013	.021	.028	.034	.039	.044	.048	.051	.054	.001
	.005	$.0^{4}62$	$.0^{2}51$	.018	.032	.044	.054	.062	.068	.073	.078	.082	.085	.005
	.010	$.0^{3}25$	.010	.029	.047	.062	.073	.082	.089	.095	.100	.104	.107	.010
	.025	$.0^{2}15$	.026	.057	.082	.100	.113	.124	.132	.139	.144	.149	.153	.025
	.05	$.0^{2}62$	.054	.099	.130	.151	.167	.179	.188	.195	.201	.207	.211	.05
	.10	.025	.117	.181	.220	.246	.265	.279	.289	.298	.304	.310	.315	.10
	.25	.172	.389	.494	.553	.591	.617	.637	.650	.661	.670	.680	.684	.25
	.50	1.00	1.50	1.71	1.82	1.89	1.94	1.98	2.00	2.03	2.04	2.05	2.07	.50
	.75	5.83	7.50	8.20	8.58	8.82	8.98	9.10	9.19	9.26	9.32	9.36	9.41	.75
	.90	39.9	49.5	53.6	55.8	57.2	58.2	58.9	59.4	59.9	60.2	60.5	60.7	.90
	.95	161	200	216	225	230	234	237	239	241	242	243	244	.95
	.975	648	800	864	900	922	937	948	957	963	969	973	977	.975
	.99	405^{1}	500^{1}	540^{1}	562^{1}	576^{1}	586^{1}	593^{1}	598^{1}	602^{1}	606^{1}	608^{1}	611^{1}	.99
	.995	162^{2}	200^{2}	216^{2}	225^{2}	231^{2}	234^{2}	237^{2}	239^{2}	241^{2}	242^{2}	243^{2}	244^{2}	.995
	.999	406^{3}	500^{3}	540^{3}	562^{3}	576^{3}	586^{3}	593^{3}	598^{3}	602^{3}	606^{3}	609^{3}	611^{3}	.999
	.9995	162^{4}	200^{4}	216^{4}	225^{4}	231^{4}	234^{4}	237^{4}	239^{4}	241^{4}	242^{4}	243^{4}	244^{4}	.9995
2	.0005	$.0^{6}50$	$.0^{3}50$	$.0^{2}42$	.011	.020	.029	.037	.044	.050	.056	.061	.065	.0005
	.001	$.0^{5}20$	$.0^{2}10$	$.0^{2}68$	.016	.027	.037	.046	.054	.061	.067	.072	.077	.001
	.005	$.0^{4}50$	$.0^{2}50$	.020	.038	.055	.069	.081	.091	.099	.106	.112	.118	.005
	.01	$.0^{3}20$	.010	.032	.056	.075	.092	.105	.116	.125	.132	.139	.144	.01
	.025	$.0^{2}13$	.026	.062	.094	.119	.138	.153	.165	.175	.183	.190	.196	.025
	.05	$.0^{2}50$	.053	.105	.144	.173	.194	.211	.224	.235	.244	.251	.257	.05
	.10	.020	.111	.183	.231	.265	.289	.307	.321	.333	.342	.350	.356	.10
	.25	.133	.333	.439	.500	.540	.568	.588	.604	.616	.626	.633	.641	.25
	.50	.667	1.00	1.13	1.21	1.25	1.28	1.30	1.32	1.33	1.34	1.35	1.36	.50
	.75	2.57	3.00	3.15	3.23	3.28	3.31	3.34	3.35	3.37	3.38	3.39	3.39	.75
	.90	8.53	9.00	9.16	9.24	9.29	9.33	9.35	9.37	9.38	9.39	9.40	9.41	.90
	.95	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.4	19.4	.95
	.975	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4	39.4	39.4	39.4	.975
	.99	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.4	99.4	.99
	.995	198	199	199	199	199	199	199	199	199	199	199	199	.995
	.999	998	999	999	999	999	999	999	999	999	999	999	999	.999
	.9995	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	200^{1}	.9995
3	.0005	$.0^{6}46$	$.0^{3}50$	$.0^{2}44$	.012	.023	.033	.043	.052	.060	.067	.074	.079	.0005
	.001	$.0^{5}19$	$.0^{2}10$	$.0^{2}71$	.018	.030	.042	.053	.063	.072	.079	.086	.093	.001
	.005	$.0^{4}46$	$.0^{2}50$	.021	.041	.060	.077	.092	.104	.115	.124	.132	.138	.005
	.01	$.0^{3}19$	.010	.034	.060	.083	.102	.118	.132	.143	.153	.161	.168	.01
	.025	$.0^{2}12$	.026	.065	.100	.129	.152	.170	.185	.197	.207	.216	.224	.025
	.05	$.0^{2}46$	.052	.108	.152	.185	.210	.230	.246	.259	.270	.279	.287	.05
	.10	.019	.109	.185	.239	.276	.304	.325	.342	.356	.367	.376	.384	.10
	.25	.122	.317	.424	.489	.531	.561	.582	.600	.613	.624	.633	.641	.25
	.50	.585	.881	1.00	1.06	1.10	1.13	1.15	1.16	1.17	1.18	1.19	1.20	.50
	.75	2.02	2.28	2.36	2.39	2.41	2.42	2.43	2.44	2.44	2.44	2.45	2.45	.75
	.90	5.54	5.46	5.39	5.34	5.31	5.28	5.27	5.25	5.24	5.23	5.22	5.22	.90
	.95	10.1	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.76	8.74	.95
	.975	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5	14.4	14.4	14.3	.975
	.99	34.1	30.8	29.5	28.7	28.2	27.9	27.7	27.5	27.3	27.2	27.1	27.1	.99
	.995	55.6	49.8	47.5	46.2	45.4	44.8	44.4	44.1	43.9	43.7	43.5	43.4	.995
	.999	167	149	141	137	135	133	132	131	130	129	129	128	.999
	.9995	266	237	225	218	214	211	209	208	207	206	204	204	.9995

Read $.0^{3}56$ as .00056, 200^{1} as 2000, 162^{4} as 1620000, etc.

Taken from (Dixon and Massey, 1957)

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